PRACTICE MIDTERM EXAM

CSC 412 - Winter 2025

University of Toronto

Exam duration: **100 minutes**

Note: The midterm will have 7 questions and so it will be shorter than this midterm practice. No calculators will be allowed during the midterm exam.

Read the following instructions carefully:

1. Exam is closed book. You can use an optional handwritten aid sheet - A4 double-sided.
2. If a question asks you to do some calculations, you must **show your work** for full credit.
3. Conceptual questions do not require long answers.
4. You will write your answers to each question in the space provided on the exam sheet. If you require additional paper, simply raise your hand.
5. After solving each question, you should write your answers immediately. Do not wait last minute to write them all at once.
6. Lastly, enjoy the problems!!!

1. Exponential families - 8pts. Probability mass function of a random variable X distributed as geometric distribution with parameter γ, with $0 < \gamma < 1$, is given as

$$\mathbb{P}(X = k) = \gamma(1-\gamma)^{k-1} \quad \text{for} \quad k = 1, 2, ...$$

(a) Show that this is a probability mass function. Hint: for $0 < p < 1$, $\sum_{k=0}^{\infty} p^k = 1/(1-p)$.
(b) Write the above distribution as an exponential family, and identify its sufficient statistics, natural parameter, and log-partition function.
(c) Assume that we observed $X_1, X_2, ..., X_n$ i.i.d. random variables from geometric distribution with an unknown parameter γ. Find the MLE for γ.

2. Maximum likelihood estimation and unnormalised models - 10 pts. Consider a model for three binary random variables (x_1, x_2, x_3),

$$p_\theta(x_1, x_2, x_3) \propto \exp\{\theta x_1 x_2 + \theta x_2 x_3\}, \qquad x_i \in \{0, 1\}.$$

1. What is the sufficient statistics of this exponential family?
2. Compute the normalizing constant $Z(\theta)$ and the derivative of $A(\theta) = \log Z(\theta)$.
3. Verify that for the sample $\{(1,1,1),(1,1,1),(1,1,0),(0,1,1),(0,1,0)\}$ the maximum likelihood is $\hat{\theta} = \ln(3)$. You will not need a calculator for this computation.
4. Compute the joint distribution $p_{\hat{\theta}}(x_1, x_2, x_3)$ corresponding to this MLE.

3. Graphical models - 16 pts. *No explanation needed, just your answers.*

(a) (4 pts) Draw the DAG corresponding to the following factorization of a joint distribution:

$$p(A, B, C, D, E) = P(A)P(B|A)P(C|A)P(D|B, C)P(E|B)$$

(b) (4 pts) Draw the Markov Random Field that corresponds to the following factorization.

$$p(A, B, C, D, E, F) \propto \phi_{A,B,F}(A, B, F)\phi_{B,C,D}(B, C, D)\phi_{D,E,F}(D, E, F)\phi_{B,D,F}(B, D, F)$$

(c) (4 pts) Write the variables that belong to the Markov blanket of node 3 in the Figure 1.

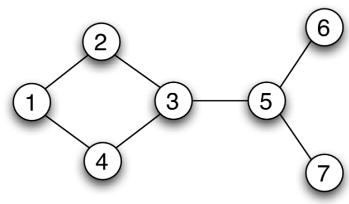


Fig 1: Simple MRF

(d) (4 pts) Belief propagation algorithm is run on a tree graph to compute the marginal of a node x.

- How many passes in which direction is sufficient to compute the marginal of x, given that we choose x to be the root?
- How many passes in which direction is sufficient to compute the marginal of x, given that we choose a root that is not the node x?

Here, the direction is either from leaves to root or from root to leaves, and a single pass refers to passing all messages pointing to one direction (either from root to leaves or from leaves to root).

4. Decision Theory - 5 pts. Imagine we are running a nuclear power plant that is undergoing a malfunction. We have two options: A) Vent the core, and B) do nothing.

Our current beliefs are that the amount of radiation in the core is uniform between 10 and 20 units, i.e.

$$\text{variant 1:} \quad R|\text{vent} \sim U(10, 20)$$

If we do nothing, there is a $X\%$ chance that no radiation will be released, and a $(100 - X)\%$ that 100 units of radiation will be released.

For what range of X would venting the core release less radiation in expectation?

5. Simple Monte Carlo - 12 pts. Imagine we have a rain prediction model that outputs samples of

$$P(R_1, R_2, \ldots, R_T|\text{measurements})$$

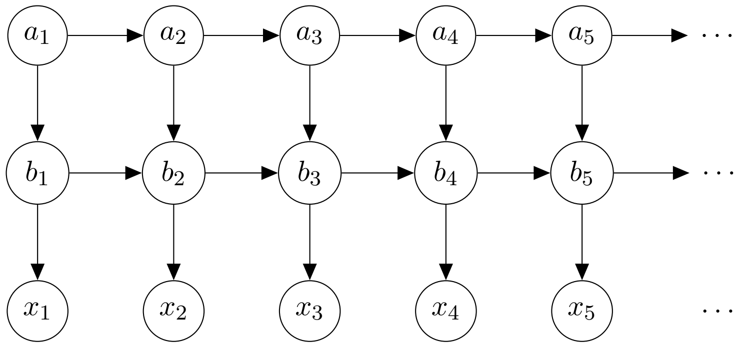
where each R_i is the predicted probability of rain i days ahead.

Given a set of N i.i.d. samples from this joint predictive distribution:

$$\begin{aligned} r_1^{(1)}, r_2^{(1)}, \ldots, r_T^{(1)} &\sim P(R_1, R_2, \ldots, R_T|\text{measurements}) \\ r_1^{(2)}, r_2^{(2)}, \ldots, r_T^{(2)} &\sim P(R_1, R_2, \ldots, R_T|\text{measurements}) \\ &\vdots \\ r_1^{(N)}, r_2^{(N)}, \ldots, r_T^{(N)} &\sim P(R_1, R_2, \ldots, R_T|\text{measurements}) \end{aligned} \tag{5.1}$$

1. **[3 points]** Write an unbiased estimator for the probability that it rains every day for the next T days. You might want to use the indicator function $\mathbb{1}(\text{statement})$ which takes value 1 if the statement is true, and 0 if it is false.
2. **[3 points]** What is the variance of this estimator as a function of N?
3. **[3 points]** Write an unbiased estimator for the probability that it rains on day 3.
4. **[3 points]** Write a consistent estimator for the probability that it rains on day 3 given that it rained on day 4.

6. Conditional Independence - 12 pts. Given the following DAG:



1. **[2 points]** Write the factorized joint distribution implied by this DAG. Don't be afraid to add extra brackets or parentheses to avoid ambiguity.

$$p(a_1, a_2, \ldots, a_T, b_1, b_2, \ldots, b_T, x_1, x_2, \ldots, x_T) =$$

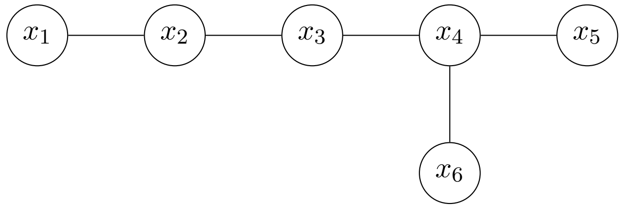
2. If each variable a_i can take one of K_a states, each variable b_i can take one of K_b states, and each variable x_i can take one of K_a states:
   - **[2 points]** How many states can this set of variables take on?
   - **[2 points]** How many parameters are required to parameterize the joint distribution $p(a_1, a_2, \ldots, a_T, b_1, b_2, \ldots, b_T, x_1, x_2, \ldots, x_T)$, again assuming the factorization given by the DAG above? Note that this factorization does not imply that the factors at each time share any parameters. Also recall that for a categorical variable with K settings, only $K - 1$ parameters are required.
3. **[1 point]** Is $x_1 \perp x_2$?
4. **[1 point]** Is $x_1 \perp x_2 | b_1$?
5. **[1 point]** Is $x_1 \perp x_2 | b_2$?
6. **[1 point]** Is $a_1 \perp a_3 | a_2$?
7. **[1 point]** Is $b_1 \perp b_3 | b_2$?
8. **[1 point]** Is $b_1 \perp b_3 | a_2, b_2$?

7. Markov chains and their stationary distributions - 15 pts. Consider a simple two-state Markov chain $x_0, x_1, x_2, \ldots$ with $x_t \in \{1, 2\}$ given by transition matrix

$$A = \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}.$$

1. Find the stationary distribution $\pi = (\pi_1, 1 - \pi_1)$ of this Markov chain. The stationary distribution is given as the solution to the vector equation $A^\top \pi = \pi$.
2. Denote $p_t = \mathbb{P}(x_t = 1)$. Find the expression for p_{t+1} in terms of p_t.
3. Show that p_t converges to π_1 as $t \to \infty$. You may want to use the fact that for $|q| < 1$ it holds that $\sum_{i=0}^{t-1} q^i = \frac{1-q^t}{1-q}$.
4. Find the exact expression for the distance $|\pi_1 - p_t|$ in terms of t and p_0 to get a quantification of how quickly the Markov chain will converge to its stationary distribution.
5. Use the Metropolis-Hastings algorithm that uses this Markov chain to generate draws from the uniform distribution on $\{1, 2\}$.

8. Belief propagation - 20 pts. Given the following graph of binary variables:



With x_4 being selected as root, having observed $\bar{x}_6 = 1$, and given the following potentials:

$\psi_{\text{even}}(x_i) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ the node potential for all x_i where i is even

$\psi_{\text{odd}}(x_i) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ The node potential for all x_i where i is odd

$\psi_{i,j}(x_i, x_j) = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ for all i, j

1. (7 points) Calculate the message from 6 to 4: $m_{6\to 4}(x_4)$
2. (7 points) Given the normalized message $m_{3\to 4}(x_3) = \begin{pmatrix} 0.55 \\ 0.45 \end{pmatrix}$ calculate $m_{4\to 5}(x_5)$
3. (6 points) Calculate $p(x_5|\bar{x}_6)$

Note: In the midterm exam the numbers will be nicer and so no calculator will be needed.

9. Miscellaneous - 10 pts.

(a) (2 pts) Describe the Markov blanket of a set of variables A. Write the variables that belong to the Markov blanket of node 2 in the Figure 2.

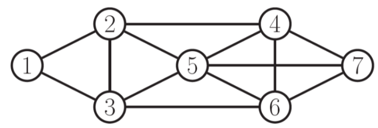


Fig 2: Simple MRF

(b) (2 pts) Identify all maximal and maximum cliques in the Figure 2.

(c) (2 pts) Describe the connection between belief propagation and variable elimination on trees.

(d) (2 pts) Compare the methods Metropolis-Hasting algorithm vs rejection sampling in terms of i) the proposal densities used ii) dependencies among the samples produced.

(e) (2 pts) In a classification problem over two classes $\mathcal{C}_1$ and $\mathcal{C}_2$, we are minimizing the misclassification error. Figure 3 shows the joint distributions. What is the decision rule that minimizes misclassification error (no derivation needed).

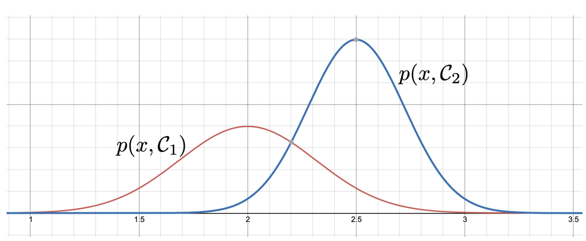


Fig 3: Decision theory