CSC 412:
Probabilistic Learning and Reasoning
Week 4 : Message Passing and Monte Carlo

Denys Linkov

University of Toronto

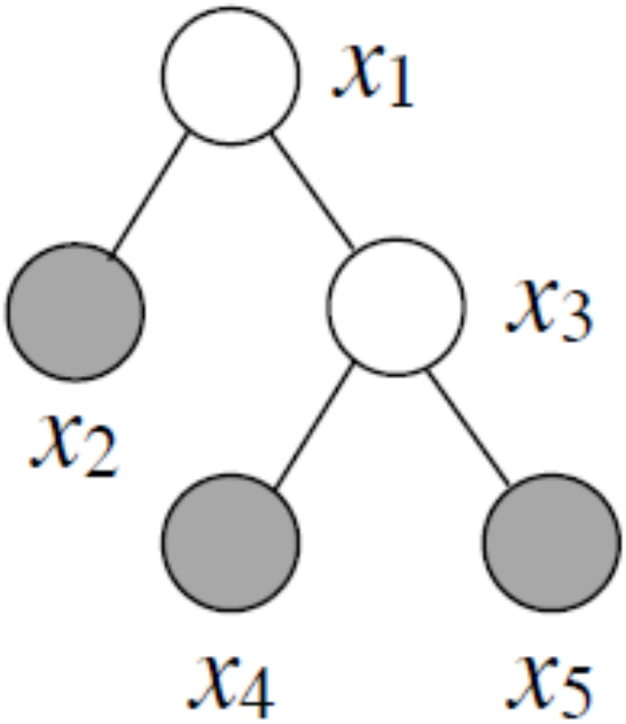
Overview

- Message passing
- Monte Carlo sampling
- Trueskill latent variable model

Variable Elimination Order and Trees

- **Last week**: we can do exact inference by variable elimination: I.e. to compute $p(A|C)$, we can marginalize $p(A, B|C)$ over every variable in B, one at a time.
- Computational cost is determined by the graph structure, and the elimination ordering.
- Determining the optimal elimination ordering is hard.
- Even if we do, the resulting marginalization might also be unreasonably costly.
- Fortunately, for trees, any elimination ordering that goes from the leaves inwards towards any root will be optimal.
- You can think of trees as just chains which sometimes branch.

Inference in Trees (MRF with no cycles)

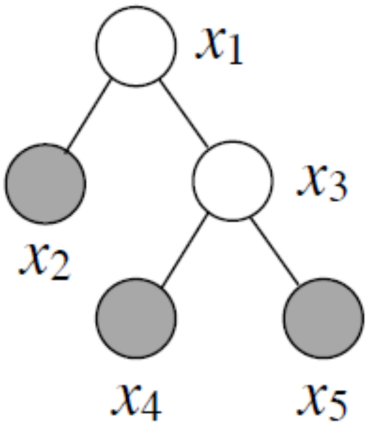


- A graph is $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of vertices (nodes) and $\mathcal{E}$ the set of edges
- For $i, j \in \mathcal{V}$, we have $(i, j) \in \mathcal{E}$ if there is an edge between the nodes i and j.
- For a node in graph $i \in \mathcal{V}$, $N(i)$ denotes the neighbors of i, i.e. $N(i) = \{j : (i, j) \in \mathcal{E}\}$.
- Shaded nodes are observed, and denoted by $\bar{x}_2, \bar{x}_4, \bar{x}_5$.

The joint distribution in the general case is

$$p(x_{1:n}) = \frac{1}{Z} \prod_{i \in \mathcal{V}} \psi(x_i) \prod_{(i,j) \in \mathcal{E}} \psi_{ij}(x_i, x_j).$$

Inference in Trees



- Joint distribution is

$$p(x_{1:n}) = \frac{1}{Z} \prod_{i \in \mathcal{V}} \psi(x_i) \prod_{(i,j) \in \mathcal{E}} \psi_{ij}(x_i, x_j).$$

- Want to compute $p(x_3 | \bar{x}_2, \bar{x}_4, \bar{x}_5)$.
- We have

$$p(x_3 | \bar{x}_2, \bar{x}_4, \bar{x}_5) \propto p(x_3, \bar{x}_2, \bar{x}_4, \bar{x}_5).$$

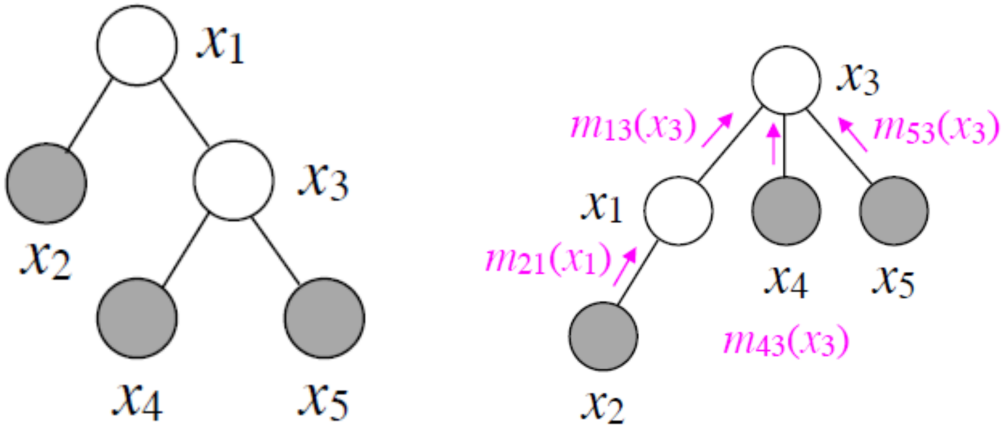
$$p(x_3 \mid \bar{x}_2, \bar{x}_4, \bar{x}_5) = \frac{1}{Z^E} \sum_{x_1} \psi_1(x_1)\psi_3(x_3)\psi_2(\bar{x}_2)\psi_4(\bar{x}_4)\psi_5(\bar{x}_5)\psi_{12}(\bar{x}_2, x_1)\psi_{34}(\bar{x}_4, x_3)\psi_{35}(\bar{x}_5, x_3)\psi_{13}(x_1, x_3)$$

- Let's write the variable elimination algorithm.

Simplifying inference on Trees

- How can we simplify relationships of a Tree Graph?
- Idea: Try to write probabilities as functions of the edges

Inference in Trees



$$
\begin{aligned}
p(x_3 \mid \bar{x}_2, \bar{x}_4, \bar{x}_5) &= \frac{1}{Z^E} \sum_{x_1} \psi_1(x_1)\psi_3(x_3)\psi_2(\bar{x}_2)\psi_4(\bar{x}_4)\psi_5(\bar{x}_5)\psi_{12}(\bar{x}_2, x_1)\psi_{34}(\bar{x}_4, x_3)\psi_{35}(\bar{x}_5, x_3)\psi_{13}(x_1, x_3) \\
&= \frac{1}{Z^E} \underbrace{\psi_4(\bar{x}_4)\psi_{34}(\bar{x}_4, x_3)}_{m_{43}(x_3)} \underbrace{\psi_5(\bar{x}_5)\psi_{35}(\bar{x}_5, x_3)}_{m_{53}(x_3)} \psi_3(x_3) \sum_{x_1} \psi_1(x_1)\psi_{13}(x_1, x_3) \underbrace{\psi_2(\bar{x}_2)\psi_{12}(\bar{x}_2, x_1)}_{m_{21}(x_1)} \\
&= \frac{1}{Z^E} \psi_3(x_3) m_{43}(x_3) m_{53}(x_3) \underbrace{\sum_{x_1} \psi_1(x_1)\psi_{13}(x_1, x_3) m_{21}(x_1)}_{m_{13}(x_3)} \\
&= \frac{1}{Z^E} \psi_3(x_3) m_{43}(x_3) m_{53}(x_3) m_{13}(x_3) = \frac{\psi_3(x_3) m_{43}(x_3) m_{53}(x_3) m_{13}(x_3)}{\sum_{x_3} \psi_3(x_3) m_{43}(x_3) m_{53}(x_3) m_{13}(x_3)}
\end{aligned}
$$

Slide credit: S. Ermon

Message Passing on Trees

We perform variable elimination from leaves to root, which is the sum product algorithm to compute all marginals. Belief propagation is a message-passing between neighboring vertices of the graph.

- The message sent from variable j to $i \in N(j)$ is

$$m_{j\to i}(x_i) = \sum_{x_j} \psi_j(x_j)\psi_{ij}(x_i, x_j) \prod_{k\in N(j)/i} m_{k\to j}(x_j)$$

  - If x_j is observed, the message is

$$m_{j\to i}(x_i) = \psi_j(\bar{x}_j)\psi_{ij}(x_i, \bar{x}_j) \prod_{k\in N(j)/i} m_{k\to j}(\bar{x}_j)$$

- Once the message passing stage is complete, we can compute our beliefs as

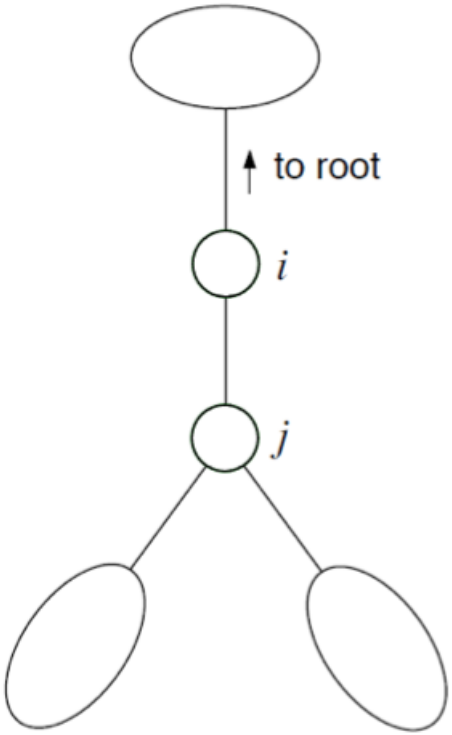
$$b(x_i) \propto \psi_i(x_i) \prod_{j\in N(i)} m_{j\to i}(x_i).$$

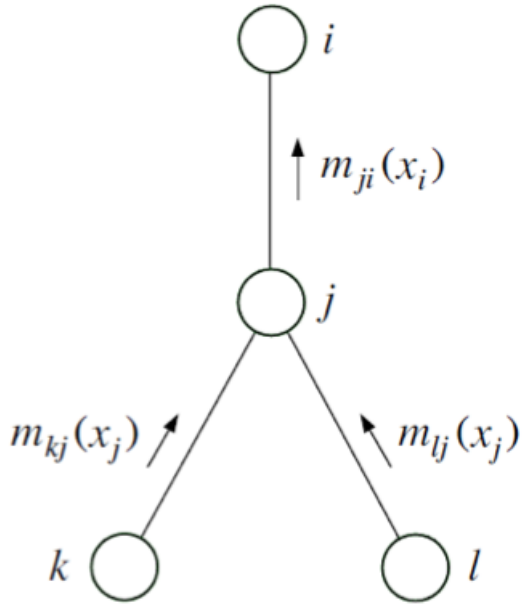
- Once normalized, beliefs are the marginals we want to compute!

Message Passing on Trees

The message sent from variable j to $i \in N(j)$ is

$$m_{j\to i}(x_i) = \sum_{x_j} \psi_j(x_j)\psi_{ij}(x_i, x_j) \prod_{k\in N(j)/i} m_{k\to j}(x_j)$$



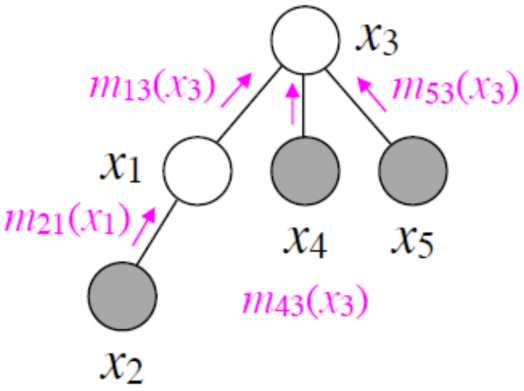


Each message $m_{j\to i}(x_i)$ is a vector with one value for each state of x_i.

Inference in Trees: Compute $p(x_3|\bar{x}_2, \bar{x}_4, \bar{x}_5)$

$$m_{j\to i}(x_i) = \sum_{x_j} \psi_j(x_j)\psi_{ij}(x_i, x_j) \prod_{k\in N(j)/i} m_{k\to j}(x_j)$$

$$b(x_i) \propto \psi_i(x_i) \prod_{j\in N(i)} m_{j\to i}(x_i).$$



- $m_{5\to 3}(x_3) = \psi_5(\bar{x}_5)\psi_{35}(x_3, \bar{x}_5)$
- $m_{2\to 1}(x_1) = \psi_2(\bar{x}_2)\psi_{12}(x_1, \bar{x}_2)$
- $m_{4\to 3}(x_3) = \psi_4(\bar{x}_4)\psi_{34}(x_3, \bar{x}_4)$
- $m_{1\to 3}(x_3) = \sum_{x_1} \psi_1(x_1)\psi_{13}(x_1, x_3)m_{2\to 1}(x_1)$
- $b(x_3) \propto \psi_3(x_3)m_{1\to 3}(x_3)m_{4\to 3}(x_3)m_{5\to 3}(x_3)$

This is the same as variable elimination, so

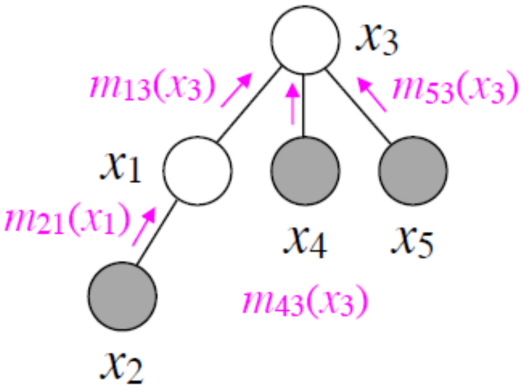
$$p(x_3|\bar{x}_2, \bar{x}_4, \bar{x}_5) = b(x_3)$$

How do we update the state of all nodes?

- We did leaf $->$ parent
- Now we can do ...

Belief Propagation on Trees

Belief Propagation Algorithm on Trees



- Choose root r arbitrarily
- Pass messages from leafs to r
- Pass messages from r to leafs
- These two passes are sufficient on trees!
- Compute beliefs (marginals)

$$b(x_i) \propto \psi_i(x_i) \prod_{j \in \mathcal{N}(i)} m_{j \to i}(x_i), \ \forall_i$$

One can compute them in two steps:

- Compute unnormalized beliefs $\tilde{b}(x_i) \propto \psi_i(x_i) \prod_{j \in \mathcal{N}(i)} m_{j \to i}(x_i)$
- Normalize them $b(x_i) = \tilde{b}(x_i) / \sum_{x_i} \tilde{b}(x_i)$.

Loopy Belief Propagation

- Example we covered is a simple case, proof link here `https://stanford.edu/~montanar/TEACHING/Stat375/handouts/bp_book.pdf`
- What if the graph (MRF) we have is not a tree and have cycles?
- Keep passing messages until convergence.
- This is called **Loopy Belief Propagation**.
- This is like when someone starts a rumour and then hears the same rumour from someone else, making them more certain it's true.
- We won't get the exact marginals, but an approximation.
- But turns out it is still very useful!

Loopy Belief Propagation

Loopy BP:

- Initialize all messages uniformly:

$$m_{i\to j}(x_j) = [1/k, ..., 1/k]^\top$$

  where k is the number of states x_j can take.
- Keep running BP updates until it "converges":

$$m_{j\to i}(x_i) = \sum_{x_j} \psi_j(x_j)\psi_{ij}(x_i, x_j) \prod_{k\in N(j)/i} m_{k\to j}(x_j)$$

  and (sometimes) normalized for stability.
- It will generally not converge, but that's generally ok.
- Compute beliefs

$$b(x_i) \propto \psi_i(x_i) \prod_{j\in\mathcal{N}(i)} m_{j\to i}(x_i).$$

This algorithm is still very useful in practice, without any theoretical guarantee (other than trees).

Sum-product vs. Max-product

- The algorithm we learned is called **sum-product BP** and approximately computes the **marginals** at each node.
- For MAP inference, we maximize over x_j instead of summing over them. This is called **max-product BP**.
- BP updates take the form

$$m_{j\to i}(x_i) = \max_{x_j} \psi_j(x_j)\psi_{ij}(x_i, x_j) \prod_{k\in N(j)\neq i} m_{k\to j}(x_j)$$

- After BP algorithm converges, the beliefs are **max-marginals**

$$b(x_i) \propto \psi_i(x_i) \prod_{j\in\mathcal{N}(i)} m_{j\to i}(x_i).$$

- MAP inference:

$$\hat{x}_i = \arg\max_{x_i} b(x_i).$$

Summary

- This algorithm is still very useful in practice, without much theoretical guarantee (other than trees).
- Loopy BP multiplies the same potentials multiple times. It is often over-confident.
- Loopy BP can oscillate, but this is generally ok.
- Loopy BP often works better if we normalize messages, and use momentum in the updates.
- The algorithm we learned is called **sum-product BP**. If we are interested in MAP inference, we can maximize over x_j instead of summing over them. This is called **max-product BP**.

Monte Carlo: Overview

- Ancestral Sampling
- Simple Monte Carlo
- Importance Sampling
- Rejection Sampling

Sampling

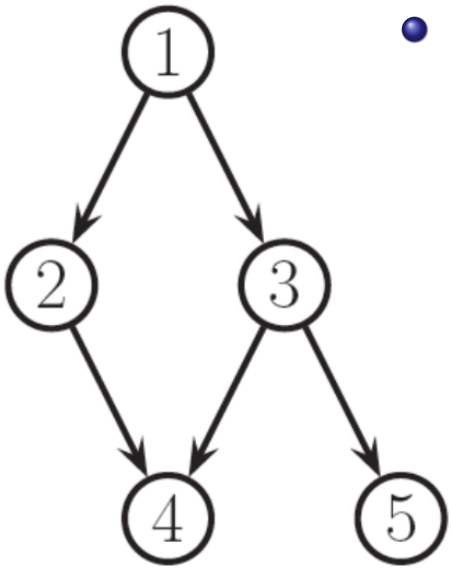
- A sample from a distribution $p(x)$ is a single realization x whose probability distribution is $p(x)$. Here, x can be high-dimensional or simply real valued.
- We assume the density from which we wish to draw samples, $p(x)$, can be evaluated to within a multiplicative constant. That is, we can evaluate a function $\tilde{p}(x)$ such that

$$p(x) = \frac{\tilde{p}(x)}{Z}.$$

Warm up: Ancestral Sampling

- Given a DAGM, and the ability to sample from each of its factors given its parents, we can sample from the joint distribution over all the nodes by **ancestral sampling**, which simply means sampling in a topoplogical order.
- At each step, sample from any conditional distribution that you haven't visited yet, whose parents have all been sampled.

Ancestral Sampling Example

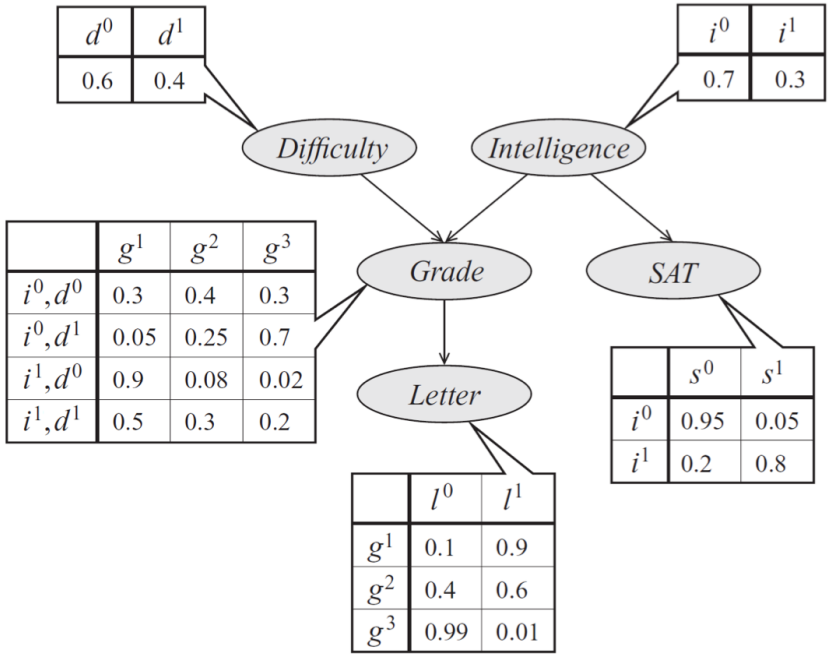


- The graph factorizes according to the local conditional probabilities

$$p(x_{1,\ldots,N}) = \prod_{i}^{N} p(x_i|\text{parents}(x_i))$$
$$= p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2,x_3)p(x_5|x_3)$$

- Start by sampling from $p(x_1)$.
- Then sample from $p(x_2|x_1)$ and $p(x_3|x_1)$.
- Then sample from $p(x_4|x_2,x_3)$.
- Finally, sample from $p(x_5|x_3)$.

Ancestral Sampling Calculations



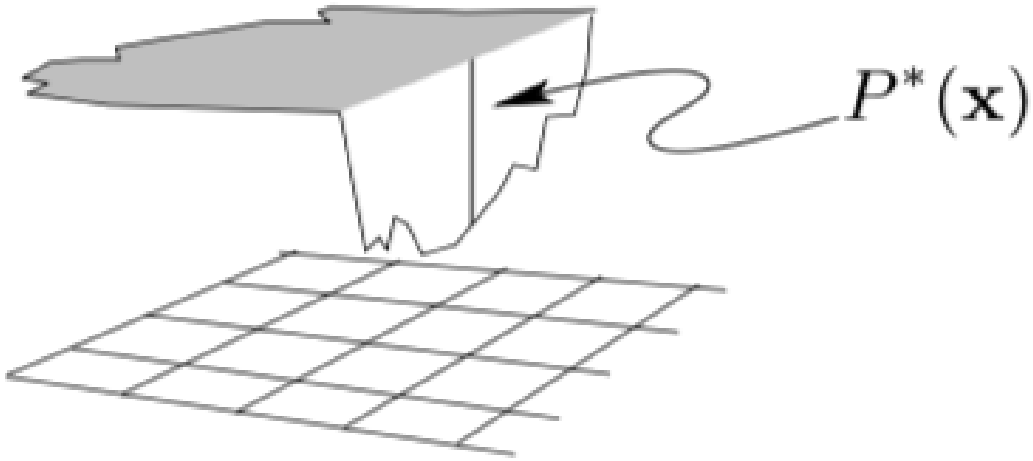
Sample	Difficulty (d)	Intelligence (i)	Grade (g)	SAT (s)	Letter (l)
1	d^0	i^1	g^1	s^1	l^1
2	d^1	i^0	g^3	s^1	l^0
3	d^0	i^1	g^1	s^1	l^0
4	d^1	i^0	g^3	s^1	l^0
5	d^0	i^1	g^1	s^1	l^0

Figure: Sampling from our graph (From CSC228)

Example: Drawing sample from a lake

Imagine the tasks of drawing random water samples from a lake and finding the average plankton concentration. Let

- $\tilde{p}(\mathbf{x})$ = the depth of the lake at $\mathbf{x} = (x, y)$
- $\phi(\mathbf{x})$ = the plankton concentration as a function of $\mathbf{x}$
- Z = the volume of the lake = $\int \tilde{p}(\mathbf{x}) d\mathbf{x}$



How would you do this?

Main objectives of sampling

We will be using Monte Carlo methods to solve one or both of the following problems.

- **Problem 1**: To generate samples $\{x^{(r)}\}_{r=1}^{R}$ from a given probability distribution $p(x)$.
- **Problem 2**: To estimate expectations of functions, $\phi(x)$, under this distribution $p(x)$

$$\Phi = \mathop{\mathbb{E}}_{x \sim p(x)}[\phi(x)] = \int \phi(x)p(x)dx$$

ϕ is called a test function.

Example

Examples of test functions $\phi(x)$:

- the **mean** of a function f under $p(x)$ by finding the expectation of the function $\phi_1(x) = f(x)$.
- the **variance** of f under $p(x)$ by finding the expectations of the functions $\phi_1(x) = f(x)$ and $\phi_2(x) = f(x)^2$

$$\phi_1(x) = f(x) \Rightarrow \Phi_1 = \mathop{\mathbb{E}}_{x \sim p(x)}[\phi_1(x)]$$

$$\phi_2(x) = f(x)^2 \Rightarrow \Phi_2 = \mathop{\mathbb{E}}_{x \sim p(x)}[\phi_2(x)]$$

$$\Rightarrow \text{var}(f(x)) = \Phi_2 - (\Phi_1)^2$$

Estimation problem

We start with the estimation problem using simple Monte Carlo:

- **Simple Monte Carlo**: Given $\{x^{(r)}\}_{r=1}^{R} \sim p(x)$ we can estimate the expectation $\mathop{\mathbb{E}}_{x\sim p(x)}[\phi(x)]$ using the estimator $\hat{\Phi}$:

$$\Phi := \mathop{\mathbb{E}}_{x\sim p(x)}[\phi(x)] \approx \frac{1}{R}\sum_{r=1}^{R}\phi(x^{(r)}) := \hat{\Phi}$$

- The fact that $\hat{\Phi}$ is a consistent estimator of Φ follows from the Law of Large Numbers (LLN).

Basic properties of Monte Carlo estimation

- **Unbiasedness:** If the vectors $\{x^{(r)}\}_{r=1}^{R}$ are generated independently from $p(x)$, then the expectation of $\hat{\Phi}$ is Φ.

$$\begin{aligned}\mathbb{E}[\hat{\Phi}] =& \mathbb{E}\left[\frac{1}{R}\sum_{r=1}^{R}\phi(x^{(r)})\right] = \frac{1}{R}\sum_{r=1}^{R}\mathbb{E}\left[\phi(x^{(r)})\right] \\ =& \frac{1}{R}\sum_{r=1}^{R}\mathop{\mathbb{E}}_{x\sim p(x)}\left[\phi(x)\right] = \frac{R}{R}\mathop{\mathbb{E}}_{x\sim p(x)}\left[\phi(x)\right] \\ =& \Phi\end{aligned}$$

Simple properties of Monte Carlo estimation

- **Variance:** As the number of samples of R increases, the variance of $\hat{\Phi}$ will decrease with rate $\frac{1}{R}$

$$\begin{aligned}
\text{var}[\hat{\Phi}] =& \text{var}\left[\frac{1}{R}\sum_{r=1}^{R}\phi(x^{(r)})\right] \\
=& \frac{1}{R^2}\text{var}\left[\sum_{r=1}^{R}\phi(x^{(r)})\right] \\
=& \frac{1}{R^2}\sum_{r=1}^{R}\text{var}\left[\phi(x^{(r)})\right] \\
=& \frac{R}{R^2}\text{var}[\phi(x)] \\
=& \frac{1}{R}\text{var}[\phi(x)]
\end{aligned}$$

Accuracy of the Monte Carlo estimate depends on the variance of ϕ.

Normalizing constant

- Assume we know the density $p(x)$ up to a multiplicative constant

$$p(x) = \frac{\tilde{p}(x)}{Z}$$

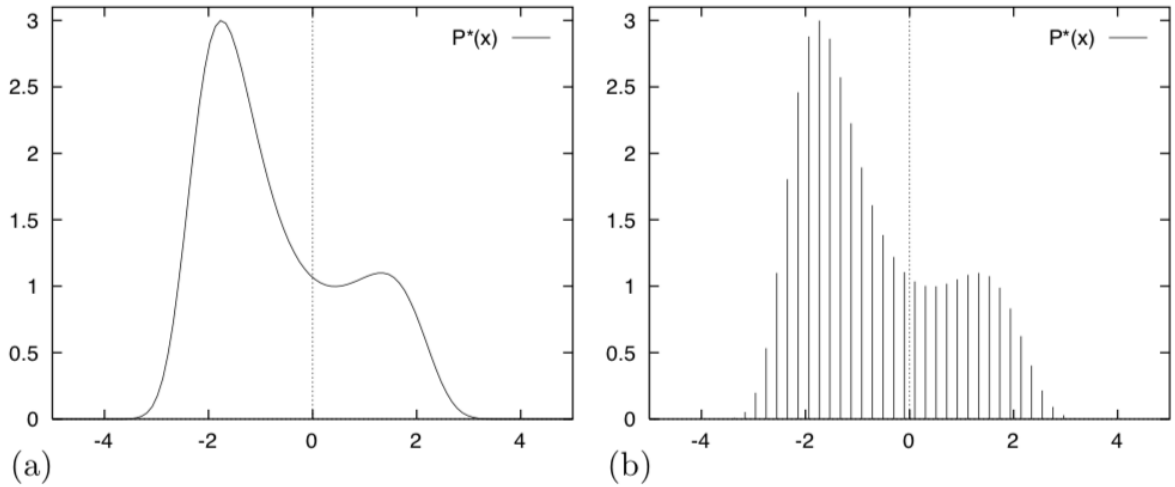
- There are two difficulties:
  - We do not generally know the normalizing constant, Z. The main diffuculty is computing it

$$Z = \int \tilde{p}(x) dx$$

    which requires computing a high-dimensional integral.
  - Even if we did know Z, the problem of drawing samples from $p(x)$ is still a challenging one, especially in high-dimensional spaces.

Bad Idea: Lattice Discretization

Imagine that we wish to draw samples from the density $p(x) = \frac{\tilde{p}(x)}{Z}$ given in figure (a).



- How to compute Z?
- We could discretize the variable x and sample from the discrete distribution (figure (b)).
- In figure (b) there are 50 uniformly spaced points in one dimension. If our system had, $D = 1000$ dimensions say, then the corresponding number of points would be $50^D = 50^{1000}$. Thus, the cost is exponential in dimension!

An analogy

Imagine the tasks of drawing random water samples from a lake and finding the average plankton concentration. Let

- $\tilde{p}(\mathbf{x})$ = the depth of the lake at $\mathbf{x} = (x, y)$
- $\phi(\mathbf{x})$ = the plankton concentration as a function of $\mathbf{x}$
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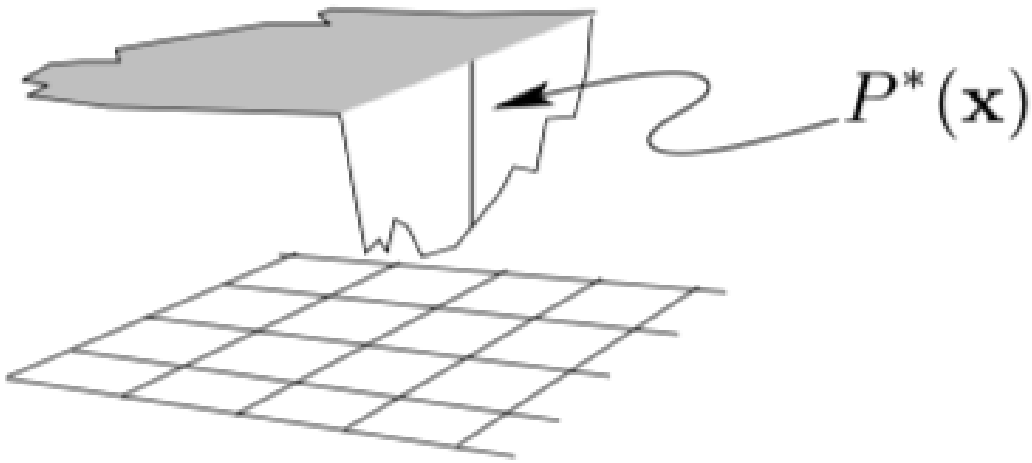


The average concentration of plankton is therefore

$$\Phi = \frac{1}{Z} \int \phi(\mathbf{x}) \tilde{p}(\mathbf{x}) d\mathbf{x}.$$

An analogy

You can take the boat to any desired location $\mathbf{x}$ on the lake, and can measure the depth, $\tilde{p}(\mathbf{x})$, and plankton concentration, $\phi(\mathbf{x})$, at that point. Therefore,

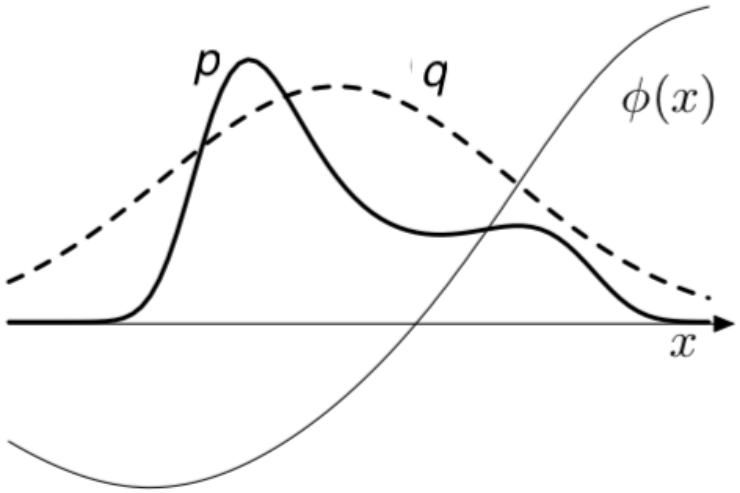
- **Problem 1** is to draw water samples at random such that each sample is equally likely to come from any point within the lake.
- **Problem 2** is to find the average plankton concentration.

A slice through a lake that includes some canyons.

- To correctly estimate Φ, our method must implicitly discover the canyons and find their volume relative to the rest of the lake.

Estimation tool: Importance Sampling

Importance sampling is a method for estimating the expectation of a function $\phi(x)$.



- The density from which we wish to draw samples, $p(x)$, can be evaluated up to normalizing constant, $\tilde{p}(x)$

$$p(x) = \frac{\tilde{p}(x)}{Z_p}$$

- There is a simpler density, $q(x)$ from which it is easy to sample from and easy to evaluate up to normalizing constant (i.e. $\tilde{q}(x)$)

$$q(x) = \frac{\tilde{q}(x)}{Z_q}$$

Estimation tool: Importance Sampling

- In importance sampling, we generate R samples from $q(x)$

$$\{x^{(r)}\}_{r=1}^{R} \sim q(x)$$

- If these points were samples from $p(x)$ then we could estimate Φ by

$$\Phi = \mathop{\mathbb{E}}_{x \sim p(x)}[\phi(x)] \approx \frac{1}{R} \sum_{r=1}^{R} \phi(x^{(r)}) = \hat{\Phi}$$

  That is, we could use a simple Monte Carlo estimator.
- But we sampled from q. We need to correct this!
- Values of x where $q(x)$ is greater than $p(x)$ will be over-represented in this estimator, and points where $q(x)$ is less than $p(x)$ will be under-represented. Thus, we introduce weights.

- Introduce weights: $\tilde{w}_r = \frac{\tilde{p}(x^{(r)})}{\tilde{q}(x^{(r)})}$ and notice that

$$\frac{1}{R}\sum_{r=1}^{R}\tilde{w}_r \approx \mathop{\mathbb{E}}_{x\sim q(x)}\left[\frac{\tilde{p}(x)}{\tilde{q}(x)}\right] = \int \frac{\tilde{p}(x)}{\tilde{q}(x)} q(x)dx = \frac{Z_p}{Z_q}$$

- Finally, we rewrite our estimator under q

$$\Phi = \int \phi(x)p(x)dx = \int \phi(x)\cdot\frac{p(x)}{q(x)}\cdot q(x)dx \approx \frac{1}{R}\sum_{r=1}^{R}\phi(x^{(r)})\frac{p(x^{(r)})}{q(x^{(r)})} = (*)$$

- However, the estimator relies on p. It can only rely on $\tilde{p}$ and $\tilde{q}$.

$$(*) = \frac{Z_q}{Z_p}\frac{1}{R}\sum_{r=1}^{R}\phi(x^{(r)})\cdot\frac{\tilde{p}(x^{(r)})}{\tilde{q}(x^{(r)})} = \frac{Z_q}{Z_p}\frac{1}{R}\sum_{r=1}^{R}\phi(x^{(r)})\cdot\tilde{w}_r$$

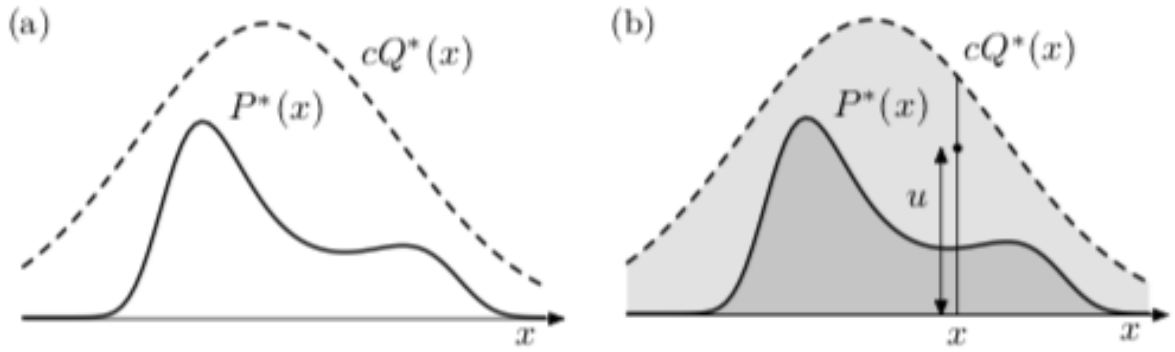
$$\approx \frac{\frac{1}{R}\sum_{r=1}^{R}\phi(x^{(r)})\cdot\tilde{w}_r}{\frac{1}{R}\sum_{r=1}^{R}\tilde{w}_r} = \sum_{r=1}^{R}\phi(x^{(r)})\cdot w_r = \hat{\Phi}_{iw}$$

where $w_r = \frac{\tilde{w}_r}{\sum_{r=1}^{R}\tilde{w}_r}$ and $\hat{\Phi}_{iw}$ is our importance weighted estimator.

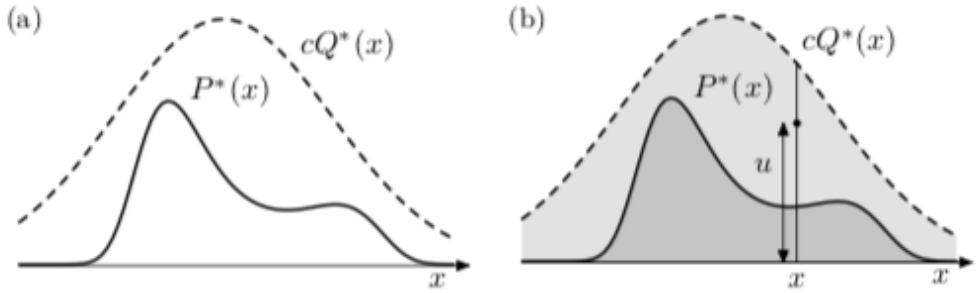
Sampling tool: Rejection sampling

- We want expectations under $p(x) = \tilde{p}(x)/Z_p$ which is a very complicated one-dimensional density.
- Assume that we have a simpler proposal density $q(x)$ which we can evaluate (within a multiplicative factor Z_q, as before), and from which we can generate samples, i.e. $\tilde{q}(x) = Z_q \cdot q(x)$.
- Further assume that we know the value of a constant c such that

$$c\tilde{q}(x) > \tilde{p}(x) \quad \forall x$$



Sampling tool: Rejection sampling



The procedure is as follows:

1. Generate two random numbers.
   - 1.1 The first, x, is generated from the proposal density $q(x)$.
   - 1.2 The second, u is generated uniformly from the interval $[0, c\tilde{q}(x)]$ (see figure (b) above: book's notation $P^* = \tilde{p}$, $Q^* = \tilde{q}$).
2. Accept or reject the sample x by comparing the value of u with the value of $\tilde{p}(x)$
   - 2.1 If $u > \tilde{p}(x)$, then x is rejected
   - 2.2 Otherwise x is accepted; x is added to our set of samples $\{x^{(r)}\}$ and the value of u discarded.

Why does rejection sampling work?

1. $x \sim q(x)$
2. $u|x \sim \text{Unif}[0, c\tilde{q}(x)]$
3. x is accepted if $u \leq \tilde{p}(x)$.

For any set A

$$\mathbb{P}_{x\sim p}(x \in A) = \int_A p(x)dx = \int \mathbf{1}_{\{x\in A\}} p(x)dx = \mathbb{E}_{x\sim p}[\mathbf{1}_{\{x\in A\}}].$$

$$\begin{aligned}
\mathbb{P}_{x\sim q}(x \in A|u \leq \tilde{p}(x)) =& \mathbb{P}_{x\sim q}(x \in A, u \leq \tilde{p}(x))/\mathbb{E}_{x\sim q}[\mathbb{P}(u \leq \tilde{p}(x)|x)] \\
=& \mathbb{E}_{x\sim q}[\mathbf{1}_{\{x\in A\}}\mathbb{P}(u \leq \tilde{p}(x)|x)]/\mathbb{E}_{x\sim q}[\frac{\tilde{p}(x)}{c\tilde{q}(x)}] \\
=& \mathbb{E}_{x\sim q}[\mathbf{1}_{\{x\in A\}}\frac{\tilde{p}(x)}{c\tilde{q}(x)}]/\frac{Z_p}{cZ_q} \\
=& \mathbb{P}_{x\sim p}(x \in A)\frac{Z_p}{cZ_q}/\frac{Z_p}{cZ_q} \\
=& \mathbb{P}_{x\sim p}(x \in A)
\end{aligned}$$

Rejection sampling in many dimensions

- In high-dimensional problems, the requirement that $c\tilde{q}(x) \geq \tilde{p}(x)$ will force c to be huge, so acceptances will be very rare.
- Finding such a value of c may be difficult too, since we don't know where the modes of $\tilde{p}$ are located nor how high they are.
- In general c grows exponentially with the dimensionality, so the acceptance rate is expected to be exponentially small in dimension

$$\text{acceptance rate} = \frac{\text{area under } \tilde{p}}{\text{area under } c\tilde{q}} = \frac{Z_p}{cZ_q}$$

Latent variables

- Latent variables are unobserved variables that govern certain properties in our probabilistic models.
- What to do when a variable z is unobserved but our model depends on it?
- If we never condition on z when in the inference problem, then we can just integrate it out.
- However, in certain cases, we are interested in the latent variables themselves, e.g. the clustering problems.
- More on latent variables when we cover Gaussian mixtures.

The TrueSkill latent variable model

- TrueSkill model is a player ranking system for competitive games.
- The goal is to infer the skill of a set of players in a competitive game, based on observing who beats who.
- In the TrueSkill model, each player has a fixed level of skill, denoted z_i.
- We initially don't know anything about anyone's skill, but we assume everyone's skill is independent (e.g. an independent Gaussian prior).
- We never get to observe the players' skills directly, which makes this a latent variable model.

TrueSkill model

- Instead, we observe the outcome of a series of matches between different players.
- For each game, the probability that player i beats player j is given by

$$p(i \text{ beats } j | z_i, z_j) = \sigma(z_i - z_j)$$

  where sigma is the logistic function: $\sigma(y) = \frac{1}{1+\exp(-y)}$.
- We can write the entire joint likelihood of a set of players and games as:

$$p(z_1, z_2, \ldots z_N, \text{game 1, game 2, .. game T})$$
$$= \left[\prod_{i=1}^{N} p(z_i)\right] \left[\prod_{\text{games}} p(i \text{ beats } j | z_i, z_j)\right]$$

Posterior

- Given the outcome of some matches, the players' skills are no longer independent, even if they've never played each other.
- Computing the **exact** posterior over even two players' skills requires integrating over all the other players' skills:

$$p(z_1, z_2|\text{game 1, game 2, ... game T})$$
$$= \int \cdots \int p(z_1, z_2, z_3 \ldots z_N|\text{games}) dz_3 \ldots dz_N$$

- **Message passing** can be used to compute **approximate** posteriors!
- More on this model in Assignment 2.

Summary

- Estimating expectations is an important problem, which is in general hard. We learned 3 sampling-based tools for this task:
  - Simple Monte Carlo
  - Importance Sampling
  - Rejection Sampling
- Next lecture, we will learn to generate samples from a particular distribution.